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Rotating Colored Black Hole in $SU(5)$ GUT

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Gravity coupled to the $SU(5)$ GUT is considered. A family of stationary axisymmetric solutions is obtained, which represents a rotating black hole with a QCD color hair as well as electromagnetic hairs.

1. INTRODUCTION

Black holes with hairs other than the total mass, angular momentum, and electric and magnetic charges have attracted much attention recently (Bowick, 1990; Bowick *et al.*, 1988; Krauss, 1990; Krauss and Wilczek, 1989; Campbell *et al.*, 1990). Krauss (1990) and Krauss and Wilczek (1989) pointed out that black holes can carry discrete hairs, and Bowick (1990), Bowick *et al.* (1988), and Campbell *et al.* (1990) discussed black holes with axion hairs.

Quite naturally, one hopes to be able to characterize all the possible hairs that can be associated with a black hole. We know that elementary particles can carry a great number of different charges. Are black holes really so different? If we consider gravity coupled to an Abelian $U(1)$ gauge field (electromagnetic field), then it is well established that a black hole can carry electric and magnetic charges. The coupled Einstein–Maxwell equations have a uniqueness theorem similar to the vacuum case—there is a unique four-parameter family of black hole solutions described by the Kerr–Newmann metric (Mazur, 1982). However, if we go to the non-Abelian case, then no analogous statements exist.

As pointed out by Bowick (1990), it is not so clear if a black hole can carry a non-Abelian charge (hair) like QCD color. In order to discuss this problem, we should consider gravity coupled to the Yang–Mills (YM) or

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Yang–Mills–Higgs (YMH) system of fields. However, the Einstein–Yang–Mills (EYM) system only describes the massless gauge fields, and the only observed massless gauge meson is the photon. All others should be massive, So, it is physically more interesting to consider the Einstein–Yang–Mills–Higgs (EYMH) system, especially a realistic EYMH system, i.e., gravity coupled to a realistic YMH unified gauge theory.

For the $SO(3)$ EYMH system, both static (Bais and Russel, 1975; Cho and Freund (1975)) and stationary (Kamata, 1982) solutions were obtained. Recently we have studied gravity coupled to a realistic, non-Abelian, spontaneously broken gauge theory—the $SU(5)$ GUT model—and obtained a family of static, spherically symmetric solutions (Yu, 1991) which may lead to a black hole with a QCD color hair.

In this paper, we present an exact rotating black hole solution for gravity coupled to the $SU(5)$ GUT model. Our solution is characterized by five physical parameters (mass M, angular momentum S, electric charge Q, magnetic charge P, and QCD color charges C^a), and represents a black hole with five hairs.

2. EQUATIONS OF MOTION AND THEIR KERR–NEWMAN TYPE SOLUTION

We consider the coupling of gravity to the $SU(5)$ GUT model in four dimensions described by the following action:

$$S=\int d^4x\,\sqrt{-g}\bigg(-\frac{R}{16\pi G}-\tfrac{1}{2}g^{\mu\rho}g^{\nu\sigma}\,\mathrm{Tr}(F_{\mu\nu}F_{\rho\sigma})+g^{\mu\nu}\,\mathrm{Tr}(D_\mu\phi D_\nu\phi)$$
$$+g^{\mu\nu}(D_\mu H)^+(D_\nu H)-V(\phi,H)\bigg) \tag{2.1}$$

with

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu-ig'[A_\mu,A_\nu],\qquad A_\mu=A^0_\mu\lambda^a,\qquad a=1,\ldots,24$$
$$D_\mu\phi=\partial_\mu\phi-ig'[A_\mu,\phi],\qquad D_\mu H=\partial_\mu H-ig'A_\mu H$$
$$V(\phi,H)=a_1\,\mathrm{Tr}\left[\left(\phi-\frac{1}{\sqrt{15}}v\right)^2\left(\phi+\frac{3}{2\sqrt{15}}v\right)^2\right]+a_2(2\,\mathrm{Tr}\,\phi^2-v^2)^2$$
$$+a_3(H^+H-\omega^2)^2+a_4H^+\left(\phi+\frac{3}{2\sqrt{15}}v\right)^2H \tag{2.2}$$
$$a_i>0,\qquad v\sim10^{14}\ \mathrm{GeV},\qquad \omega\sim10^2\ \mathrm{GeV},\qquad g'=(\tfrac{8}{3})^{1/2}e$$

where g' is the $SU(5)$ gauge coupling constant and e the charge of an electron. The group generators λ^a are chosen to satisfy $\mathrm{Tr}\, \lambda^a\lambda^b = \frac{1}{2}\delta,^{ab}\lambda^{+a} = \lambda^a$. The ϕ and H are the $\underline{24}$ rep and $\underline{5}$ rep of $SU(5)$, respectively. Their vacuum expectations are

$$\begin{aligned} &\langle\phi\rangle = v\ \mathrm{diag}(\sqrt{1/15}, \sqrt{1/15}, \sqrt{1/15}, -3/2\sqrt{15}, -3/2\sqrt{15}) \\ &\langle H\rangle = \mathrm{col}(0, 0, 0, 0, \omega) \end{aligned} \tag{2.3}$$

After the spontaneous breaking mechanism of Higgs ϕ and H, $SU(5)$ breaks down to $SU(3)_c \times U(1)_{em}$.

The equations of motion resulting from (2.1) are

$$R^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}R = 8\pi G T^{\mu\nu}$$

$$\begin{aligned} &(\sqrt{-g})^{-1}D_\nu(\sqrt{-g}\, g^{\mu\rho}g^{\nu\sigma}F_{\rho\sigma}) \\ &\quad = -ig'g^{\mu\nu}[\phi, D_\nu\phi] - ig'g^{\mu\nu}\lambda^a(H^+\lambda^a D_\nu H - (D_\nu H)^+\lambda^a H) \\ &(\sqrt{-g})^{-1}D_\mu(\sqrt{-g}\, g^{\mu\nu}D_\nu\phi) = -\frac{1}{2}\frac{\partial V(\phi, H)}{\partial\phi} \\ &(\sqrt{-g})^{-1}D_\mu(\sqrt{-g}\, g^{\mu\nu}D_\nu H) = -\frac{\partial V(\phi, H)}{\partial H^+} \end{aligned} \tag{2.4}$$

Here the space-time indices are μ, $\nu = t, r, \theta, \varphi$. With the YMH system of fields as a source of the gravitational field, the generation of nontrivial topologies by the interaction of the YMH fields and the associated gravitational field is expected. The energy-momentum tensor of the coupled EYMH system is given by

$$\begin{aligned} T^\mu{}_\nu = &-2g^{\mu\alpha}g^{\rho\beta}\,\mathrm{Tr}(F_{\alpha\beta}F_{\nu\rho}) + \tfrac{1}{2}\delta^\mu{}_\nu g^{\rho\alpha}g^{\sigma\beta}\,\mathrm{Tr}(F_{\rho\sigma}F_{\alpha\beta}) \\ &+ 2g^{\mu\rho}\,\mathrm{Tr}(D_\rho\phi D_\nu\phi) - \delta^\mu{}_\nu g^{\rho\sigma}\,\mathrm{Tr}(D_\rho\phi D_\sigma\phi) + 2g^{\mu\rho}(D_\rho H)^+(D_\nu H) \\ &- \delta^\mu{}_\nu g^{\rho\sigma}(D_\rho H)^+(D_\sigma H) + \delta^\mu{}_\nu V(\phi, H) \end{aligned} \tag{2.5}$$

In order to get the KN-type solution, we use the following stationary axisymmetric form for the space-time metric:

$$dS^2 = X\,dt^2 - Y\,dr^2 - Z\,d\theta^2 - V\,d\varphi^2 - 2W\,dt\cdot d\varphi \tag{2.6}$$

where X, Y, Z, V, and W are functions of r and θ. For the purpose of obtaining an exact rotating $SU(5)$ dyon or monopole solution, we extend the spherically symmetric Dokos and Tomaras (1980) ansatz (Yu, 1991) to the axisymmetric form as follows:

$$A_0(\mathbf{r}) = \frac{1}{g'\Sigma}\,\mathrm{diag}\left(B_1,\, B_1,\, B_2 + \frac{\hat{\mathbf{r}}\cdot\boldsymbol{\tau}}{2} B_3,\, -2(B_1+B_2)\right)$$
$$A_r = 0, \qquad A_\theta = \frac{C}{g'r}(\mathbf{T}\times\hat{\mathbf{r}})_\theta, \qquad A_\varphi = \frac{D}{g'\Sigma}(\mathbf{T}\times\hat{\mathbf{R}})_\varphi \tag{2.7}$$
$$\phi = \frac{1}{g'\Sigma}\,\mathrm{diag}(F_1,\, F_1,\, F_2 + F_3\hat{\mathbf{r}}\cdot\boldsymbol{\tau},\, -2(F_1+F_2))$$
$$H = \frac{1}{g'}\,\mathrm{col}(0,\, 0,\, 0,\, 0,\, I)$$

with

$$\mathbf{T} = \tfrac{1}{2}\,\mathrm{diag}(0,\, 0,\, \boldsymbol{\tau},\, 0), \qquad \hat{\mathbf{r}} = \mathbf{r}/|\mathbf{r}|$$
$$\hat{\mathbf{R}} = \sin\theta\cos\varphi\,\mathbf{e}_x + \sin\theta\sin\varphi\,\mathbf{e}_y + E(r,\theta)\cos\theta\,\mathbf{e}_z \tag{2.8}$$
$$\Sigma = r^2 + a^2\cos^2\theta$$

B_i, C, D, E, F_i, and I are all functions of r and θ. When the parameter a $(=S/M)$ is zero, obviously, (2.7) should reduce to the Dokos and Tomaras (1980) ansatz (Yu, 1991), i.e., we should have (Yu, 1991)

$$B_i/\Sigma = J_i(r), \qquad C = K(r) - 1, \qquad D/\Sigma = [K(r)-1]/r$$
$$E = 1, \qquad F_i/\Sigma = \phi_i(r), \qquad I = h(r) \tag{2.9}$$

For nonzero parameter a, we have obtained an exact solution of the coupled $SU(5)$ EYMH equations (2.4) as follows (see the Appendix for the verification):

$$B_1 = b_1 r = -B_2 = b_2 r, \qquad B_3 = b_3 r - a\cos\theta, \qquad (b_1 = -b_2)$$
$$C = -1$$
$$D = -r - b_3 a\cos\theta$$
$$E = \frac{r^2 + a^2 - b_3 ra\sin\theta\,\mathrm{tg}\,\theta}{r^2 + b_3 ra\cos\theta} \tag{2.10}$$
$$F_1 = \sqrt{1/15}\, vg'\Sigma, \qquad F_2 = -(1/4\sqrt{15})\, vg'\Sigma, \qquad F_3 = (5/4\sqrt{15})\, vg'\Sigma$$
$$I = \omega g'$$

and

$$X=1-\frac{G}{\Sigma}\left(2Mr-\frac{3}{8\pi}\{P^2+\tfrac{2}{3}[Q^2+(C^8)^2]\}\right)$$

$$Y=\Sigma/\Delta$$

$$Z=\Sigma$$

$$V=\left[r^2+a^2+\frac{G}{\Sigma}\left(2Mr-\frac{3}{8\pi}\{P^2+\tfrac{2}{3}[Q^2+(C^8)^2]\}\right)a^2\sin^2\theta\right]\sin^2\theta \tag{2.11}$$

$$W=-\frac{G}{\Sigma}\left(2Mr-\frac{3}{8\pi}\{P^2+\tfrac{2}{3}[Q^2+(C^8)^2]\}\right)a\sin^2\theta$$

$$\Delta=r^2+a^2-G\left(2Mr-\frac{3}{8\pi}\{P^2+\tfrac{2}{3}[Q^2+(C^8)^2]\}\right)$$

where the b_i are constants of integration. M represents the mass of the field configuration and a the angular momentum parameter, i.e., the angular momentum per unit mass. The constants P, Q, and C^8 are given by

$$P=\frac{4\pi}{2e},\qquad Q=\frac{4\pi(-b_1+b_2-b_3)}{2e},\qquad C^8=\frac{4\pi(4b_1-b_3)}{\sqrt{8}\,e} \tag{2.12}$$

Their physical meaning will be discussed in the following section.

3. DISCUSSION AND CONCLUSION

Now we investigate the long-range magnetic, electric, and color properties of our solution. Let us start by defining the electromagnetic field strength to be (Dokos and Tomaras, 1980)

$$F'_{\mu\nu}\equiv\frac{2}{g'}\,\mathrm{Tr}(F_{\mu\nu}(\mathbf{r})Q(\mathbf{r})) \tag{3.1}$$

where the electric operator is

$$Q(\mathbf{r})\underset{r\to\infty}{\to} e\,\mathrm{diag}(-\tfrac{1}{3},\,-\tfrac{1}{3},\,\tfrac{1}{3}-\tfrac{2}{3}\hat{\mathbf{r}}\cdot\boldsymbol{\tau},\,0)$$

From (2.10) and (2.7), (2.8), we have

$$
\begin{aligned}
F_{tr} &= \frac{1}{g'} \operatorname{diag}\left[\frac{\Gamma_1}{\Sigma^2}, \frac{\Gamma_1}{\Sigma^2}, \frac{\Gamma_2}{\Sigma^2} + \frac{\Gamma_3}{\Sigma^2}\frac{\hat{\mathbf{r}}\cdot\boldsymbol{\tau}}{2}, -2\left(\frac{\Gamma_1+\Gamma_2}{\Sigma^2}\right)\right] \\
F_{t\theta} &= \frac{1}{g'} \operatorname{diag}\left(\frac{\Lambda_1 a \sin\theta}{\Sigma^2}, \frac{\Lambda_1 a \sin\theta}{\Sigma^2}, \frac{a\sin\theta}{\Sigma^2}\left(\Lambda_2 + \Lambda_3 \frac{\hat{\mathbf{r}}\cdot\boldsymbol{\tau}}{2}\right), 0\right) \\
F_{\theta\varphi} &= \frac{1}{g'} \operatorname{diag}\left(0, 0, \frac{\Lambda_3}{\Sigma^2}(r^2+a^2)\sin\theta\,\frac{\hat{\mathbf{r}}\cdot\boldsymbol{\tau}}{2}, 0\right), \qquad F_{t\varphi}=0 \\
F_{\varphi r} &= \frac{1}{g'} \operatorname{diag}\left(0, 0, -\frac{\Gamma_3 a \sin^2\theta}{\Sigma^2}\frac{\hat{\mathbf{r}}\cdot\boldsymbol{\tau}}{2}, 0\right), \qquad F_{r\theta}=0
\end{aligned}
\tag{3.2}
$$

with

$$
\begin{aligned}
&\Gamma_1 = b_1(r^2 - a^2\cos^2\theta), && \Gamma_2 = b_2(r^2 - a^2\cos^2\theta) \\
&\Gamma_3 = -2ra\cos\theta + b_3(r^2 - a^2\cos^2\theta), && b_1 = -b_2 \Rightarrow \Gamma_1 + \Gamma_2 = 0 \\
&\Lambda_1 = -2b_1 ra\cos\theta, && \Lambda_2 = -2b_2 ra\cos\theta \\
&\Lambda_3 = -2b_3 ra\cos\theta - (r^2 - a^2\cos^2\theta), && b_1 = -b_2 \Rightarrow \Lambda_1 + \Lambda_2 = 0
\end{aligned}
\tag{3.3}
$$

so the ordinary electric field $\mathbf{E}(\mathbf{r})$ and magnetic field $\mathbf{B}(\mathbf{r})$

$$
\begin{aligned}
E_i(\mathbf{r}) &\equiv \frac{2}{g'} \operatorname{Tr}(F_{0i}(\mathbf{r})Q(\mathbf{r})) \\
B_i(\mathbf{r}) &\equiv \frac{1}{g'} \operatorname{Tr}(\varepsilon_{ijk}F_{jk}(\mathbf{r})Q(\mathbf{r}))
\end{aligned}
\tag{3.4}
$$

can be calculated as follows:

$$
\begin{aligned}
(E_r, E_\theta, E_\varphi) &= (F'_{rt}, F'_{\theta t}, F'_{\varphi t}) \\
&\underset{r\to\infty}{\rightarrow} \left(\frac{-\Gamma_1+\Gamma_2-\Gamma_3}{2e\Sigma^2}, \frac{a\sin\theta}{2e\Sigma^2}(-\Lambda_1+\Lambda_2-\Lambda_3), 0\right) \\
(B_r, B_\theta, B_\varphi) &= (F'_{\theta\varphi}, F'_{\varphi r}, F'_{r\theta}) \\
&\underset{r\to\infty}{\rightarrow} \left(\frac{-1}{2e\Sigma^2}\Lambda_3(r^2+a^2)\sin\theta, \Gamma_3 a\sin^2\theta, 0\right)
\end{aligned}
\tag{3.5}
$$

which, when expresssed in the asymptotic rest frame (Misner *et al.*, 1973), have the following radial orthonormal components:

$$\begin{aligned}(E_{r'}, E_{\theta'}, E_{\varphi'}) \underset{r\to\infty}{\rightsquigarrow} \left(\frac{-\Gamma_1+\Gamma_2-\Gamma_3}{2e\Sigma^2}, 0, 0\right) \to \left(\frac{-b_1+b_2-b_3}{2er^2}, 0, 0\right)\\ = \left(\frac{4\pi(-b_1+b_2-b_3)}{2e}\frac{1}{4\pi r^2}, 0, 0\right)\\ (B_{r'}, B_{\theta'}, B_{\varphi'}) \underset{r\to\infty}{\rightsquigarrow} \left(\frac{-\Lambda_3}{2e\Sigma^2}, 0, 0\right) \to \left(\frac{1}{2er^2}, 0, 0\right)\\ = \left(\frac{4\pi}{2e}\frac{1}{4\pi r^2}, 0, 0\right)\end{aligned} \tag{3.6}$$

We now proceed to the investigation of the color properties of our solution. By analogy, we can define the color-electric and color-magnetic fields, respectively, as

$$\begin{aligned}E_i^a(\mathbf{r}) \equiv 2\,\mathrm{Tr}(F_{0i}(\mathbf{r})\lambda^a(\mathbf{r}))\\ B_i^a(\mathbf{r}) \equiv \mathrm{Tr}(\varepsilon_{ijk}F_{jk}(\mathbf{r})\lambda^a(\mathbf{r})), \qquad a=1,\ldots,8\end{aligned} \tag{3.7}$$

where $\lambda^a(\mathbf{r})$ with a ranging from 1 to 8 are the generators of the $SU(3)_c$ subgroup. By looking at what happens along the $+z'$ axis at infinity, we conclude

$$\begin{aligned}(E_{r'}^a, E_{\theta'}^a, E_{\varphi'}^a) \underset{r\to\infty}{\rightsquigarrow} \left(\frac{4\pi}{\sqrt{8}\,e}(4b_1-b_3)\frac{\delta^{a8}}{4\pi r^2}, 0, 0\right) = \left(\frac{C^a}{4\pi r^2}, 0, 0\right)\\ (B_{r'}^a, B_{\theta'}^a, B_{\varphi'}^a) \underset{r\to\infty}{\rightsquigarrow} \left(\frac{1}{\sqrt{3}}\frac{\delta^{a\delta}}{g'r^2}, 0, 0\right)\end{aligned} \tag{3.8}$$

Consequently, we can see from (3.6) and (3.8) that the constants P, Q, and C^8, respectively, have the physical meaning of the magnetic, electric, and color charge carried by the field configuration of our solution.

From the metric (2.11), we can show in Boyer–Lindquist coordinates that there is an event horizon at

$$r_H = GM + \left((GM)^2 - a^2 - \frac{3G}{\delta\pi}\{P^2 + \tfrac{2}{3}[Q^2 + (C^8)^2]\}\right)^{1/2} \tag{3.9}$$

So our solution represents a QCD colored, electromagnetically charged, rotating black hole provided that

$$a^2+\frac{3G}{8\pi}\{P^2+\tfrac{2}{3}[Q^2+(C^8)^2]\}\leq (GM)^2 \tag{3.10}$$

Our black hole is smaller than the Kerr–Newman black hole. The black hole shrinks as the number of physical conserved quantities increases.

Now we would like to end the discussion part of this section by pointing out that the metric (17) in Yu (1991) can be written

$$\begin{aligned} ds^2 &= \left(1-\frac{2GM}{r}+\frac{3G}{8\pi r^2}\{P^2+\tfrac{2}{3}[Q^2+(C^8)^2]\}\right)dt^2 \\ &\quad -\left(1-\frac{2GM}{r}+\frac{3G}{8\pi r^2}\{P^2+\tfrac{2}{3}[Q^2+(C^8)^2]\}\right)^{-1} dr^2 \\ &\quad -r^2(d\theta^2+\sin^2\theta\, d\varphi^2) \end{aligned} \tag{3.11}$$

Thus the solution given in Yu (1991) for the coupled $SU(5)$ EYMH system can lead to a QCD colored, electromagnetically charged, static black hole if the following relation is satisfied:

$$(GM)^2\geq \frac{3G}{8r}\{P^2+\tfrac{2}{3}[Q^2+(C^8)^2]\} \tag{3.12}$$

In conclusion, we have presented an exact stationary axisymmetric solution for the coupled $SU(5)$ EYMH system. Our solution gives the gravitation and gauge fields of a ring of mass M, electric charge Q, magnetic charge P, and QCD color charge C^8 rotating about its axis of symmetry. It is a black hole solution if (3.10) is satisfied, and reduces to the result given in Yu (1991) in the case $a=0$. Our solution leads to a QCD colored, electromagnetically charged, rotating black hole, i.e., a black hole with five hairs.

APPENDIX

Using the metric (2.6) and equations (2.10) and (2.11), we can compute the main nonzero components of the Einstein tensor $E^\mu{}_\nu = R^\mu{}_\nu - \frac{1}{2}\delta^\mu{}_\nu R$

as follows:

$$E^t{}_t=\frac{1}{4Y\rho}\bigg[2X\,\partial_r^2V+\partial_rV\,\partial_rX+2W\,\partial_r^2W+(\partial_rW)^2$$

$$-(X\,\partial_rV+W\,\partial_rW)\,\partial_r\left(\ln\frac{\rho Y}{Z}\right)\bigg]$$

$$+\frac{1}{4Z\rho}\bigg[2X\,\partial_\theta^2V+\partial_\theta V\,\partial_\theta X+2W\,\partial_\theta^2W+(\partial_\theta W)^2$$

$$-(X\,\partial_\theta V+W\,\partial_\theta W)\,\partial_\theta\left(\ln\frac{\rho Z}{Y}\right)\bigg]$$

$$+\frac{1}{4YZ}[2\,\partial_r^2Z+2\,\partial_\theta^2Y-\partial_rZ\,\partial_r(\ln\,YZ)-\partial_\theta Y\,\partial_\theta(\ln\,YZ)]$$

$$E^t{}_\varphi=-\frac{1}{4Y\rho}\bigg[2\,\partial_r(W\,\partial_rV-V\,\partial_rW)-(W\,\partial_rV-V\,\partial_rW)\,\partial_r\left(\ln\frac{\rho Y}{Z}\right)\bigg]$$

$$-\frac{1}{4Z\rho}\bigg[2\,\partial_\theta(W\,\partial_\theta V-V\,\partial_\theta W)-(W\,\partial_\theta V-V\,\partial_\theta W)\,\partial_\theta\left(\ln\frac{\rho Z}{Y}\right)\bigg]$$

$$E^r{}_r=\frac{1}{4Y\rho}[\partial_r\rho\,\partial_r(\ln\,Z)+\partial_rX\,\partial_rV+(\partial_rW)^2]$$

$$+\frac{1}{4Z\rho}[2\,\partial_\theta^2\rho-\partial_\theta\rho\,\partial_\theta(\ln\,\rho Z)-\partial_\theta X\,\partial_\theta V-(\partial_\theta W)^2]$$

$$E^r{}_\theta=-\frac{1}{4Y\rho}[2\,\partial_r\,\partial_\theta\rho-\partial_r\rho\,\partial_\theta(\ln\,\rho Y)-\partial_r(\ln\,Z)\,\partial_\theta\rho$$

$$-(\partial_rX\,\partial_\theta V+\partial_\theta X\,\partial_rV+2\,\partial_rW\,\partial_\theta W)]$$

$$E^\theta{}_\theta=\frac{1}{4Y\rho}[2\,\partial_r^2\rho-\partial_r\rho\,\partial_r(\ln\,\rho Y)-\partial_rX\,\partial_rV-(\partial_rW)^2]$$

$$+\frac{1}{4Z\rho}[\partial_\theta\rho\,\partial_\theta(\ln\,Y)+\partial_\theta X\,\partial_\theta V+(\partial_\theta W)^2]$$

$$E^\varphi{}_\varphi=\frac{1}{4Y\rho}\bigg[2V\,\partial_r^2X+\partial_rX\,\partial_rV+2W\,\partial_r^2W$$

$$+(\partial_rW)^2-(V\,\partial_rX+W\,\partial_rW)\,\partial_r\left(\ln\frac{\rho Y}{Z}\right)\bigg]$$

$$+\frac{1}{4Z\rho}\left[2V\,\partial_\theta^2 X+\partial_\theta X\,\partial_\theta V+2W\,\partial_\theta^2 W\right.$$

$$\left.+(\partial_\theta W)^2-(V\,\partial_\theta X+W\,\partial_\theta W)\,\partial_\theta\left(\ln\frac{\rho Z}{Y}\right)\right]$$

$$+\frac{1}{4YZ}[2\,\partial_r^2 Z+2\,\partial_\theta^2 Y-\partial_r Z\,\partial_r(\ln\,YZ)-\partial_\theta Y\,\partial_\theta(\ln\,YZ)]$$

where $\rho=XV+W^2$ and $\sqrt{-g}=(\rho YZ)^{1/2}$. From (2.7), (2.8), and (2.10), after a tedious calculation, we obtain:

$$F_{tr}=\frac{1}{g'\Sigma^2}\,\mathrm{diag}\left(\Gamma_1,\,\Gamma_1,\,\Gamma_2+\Gamma_3\frac{\hat{\mathbf{r}}\cdot\boldsymbol{\tau}}{2},\,-2(\Gamma_1+\Gamma_2)\right)$$

$$F_{t\theta}=\frac{a\sin\theta}{g'\Sigma^2}\,\mathrm{diag}\left(\Lambda_1,\,\Lambda_1,\,\Lambda_2+\Lambda_3\frac{\hat{\mathbf{r}}\cdot\boldsymbol{\tau}}{2},\,0\right)$$

$$F_{t\varphi}=0$$

$$F_{\theta\varphi}=\frac{1}{g'\Sigma^2}\,\mathrm{diag}\left(0,\,0,\,\Lambda_3(r^2+a^2)\sin\theta\frac{\hat{\mathbf{r}}\cdot\boldsymbol{\tau}}{2},\,0\right)$$

$$F_{\varphi r}=\frac{1}{g'\Sigma^2}\,\mathrm{diag}\left(0,\,0,\,-\Gamma_3 a\sin^2\theta\frac{\hat{\mathbf{r}}\cdot\boldsymbol{\tau}}{2},\,0\right)$$

$$F_{r\theta}=0$$

$$D_\mu\phi=0,\qquad \mu=t,\,r,\,\theta,\,\varphi$$

$$D_\mu H=0,\qquad \mu=t,\,r,\,\theta,\,\varphi$$

(in deriving $D_\mu H=0$, we have used the relation $B_1=-B_2$), where

$$\Gamma_1=b_1(r^2-a^2\cos^2\theta),\qquad \Gamma_2=b_2(r^2-a^2\cos^2\theta)$$

$$\Gamma_3=-2ra\cos\theta+b_3(r^2-a^2\cos^2\theta),\qquad b_1=-b_2\ \Rightarrow\ \Gamma_1+\Gamma_2=0$$

$$\Lambda_1=-2b_1 ra\cos\theta,\qquad \Lambda_2=-2b_2 ra\cos\theta$$

$$\Lambda_3=-2b_3 ra\cos\theta-(r^2-a^2\cos^2\theta),\qquad b_1=-b_2\ \Rightarrow\ \Lambda_1+\Lambda_2=0$$

To get the explicit forms for the energy-momentum tensor, we use the contravariant metric tensor $g^{\mu\nu}$, which, according to (2.6), is given by

$$g^{tt}=\frac{V}{\rho}, \qquad g^{rr}=-\frac{1}{Y}, \qquad g^{\theta\theta}=-\frac{1}{Z}$$

$$g^{\varphi\varphi}=-\frac{X}{\rho}, \qquad g^{t\varphi}=-\frac{W}{\rho}, \qquad \rho=XV+W^2$$

Inserting all the above results into (2.2), we find the nonzero components of $T^{\mu}{}_{\nu}$:

$$T^{t}{}_{t}=\frac{3}{64\pi^2}\{P^2+\tfrac{2}{3}[Q^2+(C^8)^2]\}\frac{1}{\Sigma^3}[r^2+a^2(1+\sin^2\theta)]=-T^{\varphi}{}_{\varphi}$$

$$T^{t}{}_{\varphi}=-\frac{3}{32\pi^2}\{P^2+\tfrac{2}{3}[Q^2+(C^8)^2]\}\frac{1}{\Sigma^3}(r^2+a^2)a\sin^2\theta$$

$$T^{\varphi}{}_{t}=\frac{3}{32\pi^2}\frac{a}{\Sigma^3}\{P^2+\tfrac{2}{3}[Q^2+(C^8)^2]\}$$

$$T^{r}{}_{r}=\frac{3}{64\pi^2}\{P^2+\tfrac{2}{3}[Q^2+(C^8)^2]\}\frac{1}{\Sigma^3}=-T^{\theta}{}_{\theta}$$

Making use of all these expressions, we have checked that (2.10) and (2.11) are really a solution of the coupled EYMH equations of motion (2.4).

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